MATHEMATICS

Mob. : 9470844028
9546359990

AIM POINTS
MATHEMATICS
DIR. FIROZ AHMAD

Ram Rajya More, Siwan

XIth, XIIth, TARGET IIT-JEE (MAIN + ADVANCE) & COMPETITIVE EXAM. FOR XII (PQRS)

VECTOR OR CROSS PRODUCT

& Their Properties

CONTENTS

THINGS TO REMEMBER

1. If $\vec{a}.\vec{b}$ are two vectors inclined at an angle θ, then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}| \sin\theta\,\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{n}$ form a right handed system.

2. $\vec{a}\times\vec{b}=\vec{0}$ iff $\vec{a}$ and $\vec{b}$ are parallel.

3. $\vec{a}\times\vec{b}=$ Vector area of the paralleogram having two adjacent sides as $\vec{a}$ and $\vec{b}$.

4. Unit vectors perpendicular to the plane of $\vec{a}$ and $\vec{b}$ are $\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

5. $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

6. (i) $m(\vec{a}\times\vec{b})\ m\vec{a}\times\vec{b} = \vec{a}\times m\vec{b}$, for any scalar m

   (ii) $m\vec{a}\times n\vec{b} = mn\ (\vec{a}\times\vec{b}) = mn\vec{a}\times\vec{b} = \vec{a}\times mn\vec{b}$ for scalars m, n.

7. $\vec{a}\times\vec{b}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$, where $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

8. For any two vectors $\vec{a}$ and $\vec{b}$, we have $|\vec{a}\times\vec{b}|^2 + (\vec{a}.\vec{b})^2 = |\vec{a}|^2\,|\vec{b}|^2$

9. For any vector $\vec{a}$, we have $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = 2\,|\vec{a}|^2$

10. (i) Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| = \frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}| = \frac{1}{2}|\overrightarrow{CB}\times\overrightarrow{CA}|$

    (ii) Area of a plane convex quadrilateral ABCD $= \frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$, where AC and BD are diagonal.

11. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B, C of ΔABC, then

    Area of $\Delta ABC = \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

    Length of the perpendicular from C on AB $= \dfrac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{a}-\vec{b}|}$

    Length of the perpendicular from A on BC $= \dfrac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{b}-\vec{c}|}$

    Length of the perpendicular from B on AC $= \dfrac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{c}-\vec{a}|}$

EXERCISE-1

1. Vector product is not commutative i.e. if $\vec{a}$ and $\vec{b}$ are any two vectors, then, $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$.
2. If $\vec{a},\vec{b}$ are two vectors and m is a scalar, then $m\vec{a}\times\vec{b} = m(\vec{a}\times\vec{b}) = \vec{a}\times m\vec{b}$
3. Distributivity of vector product over vector addition. Let $\vec{a},\vec{b},\vec{c}$ be any three vectors. Then,

   (i) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

   (ii) $(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a}+\vec{c}\times\vec{a}$
4. Find $\vec{a}\times\vec{b}$, if $\vec{a} = 2\hat{i}+\hat{k}$ and $\vec{b} = \hat{i}+\hat{j}+\hat{k}$.
5. Find the magnitude of $\vec{a}$ given by $\vec{a} = (\hat{i}+3\hat{j}-2\hat{k})\times(-\hat{i}+3\hat{k})$
6. Find a unit vector perpendicular to both the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $\hat{i}+2\hat{j}-\hat{k}$.
7. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$.
8. Find a vector of magnitude 9, which is perpendicular to both the vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$.
9. Find a unit vector perpendicular to the plane ABC where, A,B,C are the points (3, –1, 2), (1, –1, –3), (4, –3, 1) respectively.
10. Find the area of the parallelogram determined by the vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$.
11. Show that the area of a parallelogram having diagonals $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$ is $5\sqrt{3}$.
12. Given $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 12$, find $|\vec{a}\times\vec{b}|$.
13. Find the area of the triangle whose vertices are A(3, –1, 2), B(1, –1, –3) and C(4, –1, 1).
14. Show that $(\vec{a}\times\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$
15. If $\vec{a},\vec{b},\vec{c}$ are the position vectors of the vertices A, B, C of a triangle ABC, show that the area of triangle ABC is $\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$.
16. Prove that the points A, B and C with position vectors $\vec{a},\vec{b}$ and $\vec{c}$ respectively are collinear if and only if $|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}| = \vec{0}$.
17. Show that distance of the point $\vec{c}$ from the line joining $\vec{a}$ and $\vec{b}$ is

$$\frac{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|}{|\vec{b}-\vec{a}|}$$

18. For any three vectors $\vec{a}, \vec{b}, \vec{c}$. Show that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$.

19. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$, where $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.

20. Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, prove that $\vec{a} = \pm 2 (\vec{b} \times \vec{c})$.

21. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

22. Prove that the normal to the plane containing three points whose position vectors are $\vec{a}, \vec{b}, \vec{c}$ lies in the direction $\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}$.

23. If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, $\vec{a} \neq \vec{0}$, then show that $\vec{b} = \vec{c}$.

24. Prove that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$ and interpret it geometrically.

25. For any vector $\vec{a}$, prove that $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2|\vec{a}|^2$.

26. Let $\overline{OA} = \vec{a}, \overline{OB} = 10\vec{a} + 2\vec{b}$, and $\overline{OC} = \vec{b}$ where O is origin. Let promote the area of the quadrilateral OABC and q denote the area of the paralleogram with OA and OC as adjacent sides. Prove that p = 6q.

27. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{j} - \hat{k}$ and $\vec{c} = 7\hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c} \cdot \vec{d} = 1$.

28. Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.

29. For any two vectors $\vec{a}$ and $\vec{b}$, show that :

$$\left(1 + |\vec{a}|^2\right)\left(1 + |\vec{b}|^2\right) = |(1 - \vec{a}.\vec{b})|^2 + |\vec{a} + \vec{b} + (\vec{a} \times \vec{b})|^2$$

30. ABCD is a quadrilateral such that $\overline{AB} = \vec{b}$, $\overline{AD} = \vec{d}$, $\overline{AC} = m\vec{b} + p\vec{d}$. Show that the area of the quadrilateral ABCD is $\frac{1}{2}|m + p||\vec{b} \times \vec{d}|$

31. If A,B,C,D be any four points in space, prove that $|\overline{AB} \times \overline{CD} + \overline{BC} \times \overline{AD} + \overline{CA} \times \overline{BD}| = 4$ (Area of trianble ABC).

32. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{j} - \hat{k}$ are given vectors, then find a vector $\vec{b}$ satisfying the equations $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a} \cdot \vec{b} = 3$.

33. If $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = -\hat{i} + 3\hat{k}$, find $|\vec{a} \times \vec{b}|$

34. (i) If $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$, find the value of $|\vec{a} \times \vec{b}|$.

(ii) If $\vec{a} = 2\hat{i} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$, find the value of $\vec{a} \times \vec{b}$.

35. (i) Find a unit vector perpendicular to both the vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$

(ii) Find a unit vector perpendicular to the plane containing the vectors $\vec{a}=2\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$

36. Find the magnitude of $\vec{a}=(3\hat{i}+4\hat{j})\times(\hat{i}+\hat{j}-\hat{k})$

37. If $\vec{a}=4\hat{i}+3\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{k}$, then find $|2\vec{b}\times\vec{a}|$.

38. If $\vec{a}=3\hat{i}-\hat{j}-2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$, find $(\vec{a}+2\vec{b})\times(\vec{a}-2\vec{b})$.

39. Find the area of the parallelogram determined by the vectors :

(i) $2\hat{i}$ and $3\hat{j}$

(ii) $2\hat{i}+\hat{j}+3\hat{k}$ and $\hat{i}-\hat{j}$

(iii) $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$

(iv) $\hat{i}-3\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$.

40. If $|\vec{a}|=2$, $|\vec{b}|=7$ and $\vec{a}\times\vec{b}=3\hat{i}+2\hat{j}+6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$.

41. If $|\vec{a}|=\sqrt{26}$, $|\vec{b}|=7$ and $|\vec{a}\times\vec{b}|=35$ $\vec{a}\cdot\vec{b}$.

42. Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$, $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}\cdot\vec{d}=15$.

43. Using vectors find the area of the triangle with vertices, A(2, 3, 5), B (3, 5, 8) and C(2, 7, 8).

44. Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$, $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}\cdot\vec{d}=15$.

45. Given that $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?

46. If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$. Is the converse true ? Justify your answer with an example.

47. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then verify that $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.

EXERCISE-2

1. For any two vectors $\vec{a}$ and $\vec{b}$ write the value of $(\vec{a}\cdot\vec{b})^2+|\vec{a}\times\vec{b}|^2$ in terms of their magnitudes.

2. If $\vec{a}$ and $\vec{b}$ are two vectors of magnitudes 3 and $\frac{\sqrt{2}}{3}$ respectively such that $\vec{a}\times\vec{b}$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$.

3. If $|\vec{a}| = 10, |\vec{b}| = 2$ and $|\vec{a} \times \vec{b}| = 16$, find $\vec{a} \cdot \vec{b}$.

4. For any three vectors $\vec{a} \times (\vec{b}+\vec{c})+\vec{b} \times (\vec{c}+\vec{a})+\vec{c}(\vec{a}+\vec{b})$.

5. Write the value of $\hat{i} \times (\hat{j} \times \hat{k})$.

6. If $\vec{a}=3\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}-\hat{k}$, then find $(\vec{a} \times \vec{b})\vec{a}$.

7. Write a unit vector perpendicular to $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$.

8. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then write the value of $|\vec{r} \times \hat{i}|^2$.

9. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $\vec{a} \times \vec{b}$ is also a unit vector, find the angle between $\vec{a}$ and $\vec{b}$.

10. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$, write the angle between $\vec{a}$ and $\vec{b}$.

11. If $\vec{a}$ and $\vec{b}$ are unit vectors, then write the value of $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2$.

12. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and when $|\vec{a} \times \vec{b}| = \sqrt{3}$.

13. Vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}| = \sqrt{3}, |\vec{b}| = \frac{2}{3}$ and $(\vec{a} \times \vec{b})$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$.

14. Find λ, if $(2\hat{i}+6\hat{j}+14\hat{k}) \times (\hat{i}-\lambda\hat{j}+7\hat{k}) = \vec{0}$.

EXERCISE-3

1. If $\vec{a}$ is any vector, then $(\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2 =$

   (a) $\vec{a}^2$ (b) $2\vec{a}^2$ (c) $3\vec{a}^2$ 4(d) $\vec{a}^2$

2. The vector $\vec{b} = 3\hat{i}+4\hat{k}$ is to be writen as the sum of a vector $\vec{\alpha}$ parallel to $\vec{a}=\hat{i}+\hat{j}$ and a vector $\vec{\beta}$ perpendicular to $\vec{a}$. Then $\vec{\alpha}$ =

   (a) $\frac{3}{2}(\hat{i}+\hat{j})$ (b) $\frac{2}{3}(\hat{i}+\hat{j})$ (c) $\frac{1}{2}(\hat{i}+\hat{j})$ (d) $\frac{1}{3}(\hat{i}+\hat{j})$

3. The value of $\hat{i} \cdot (\hat{j} \times \hat{k})+\hat{j} \cdot (\hat{i} \times \hat{k})+\hat{k} \cdot (\hat{i} \times \hat{j})$, is

   (a) 0 (b) –1 (c) 1 (d) 3

4. If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to

   (a) 0 (b) $\pi/4$ (c) $\pi/2$ (d) π